Iterated extended Kalman filter for airborne electromagnetic data inversion

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SUMMARY

Iterated extended Kalman filter is a tool in the theory of optimal estimation used for nonlinear problems. It minimizes variance of the estimation error in terms of probabilistic approach. Despite the special terminology, the Kalman filter algorithm minimizes the objective function, representing the squared difference between the measured vector and the calculated one for the parameters of selected model. It works like the least squares method – a conventional method for airborne electromagnetic data inversion. In this paper I describe the essence of the Kalman approach to solving inverse problems. I show, how one-dimensional inversion with vertical and lateral constraints can be performed in terms Kalman filter. The described algorithm takes into account the measurement noise, which is specified as the dispersion of signals in the corresponding measurement channels at high altitude. Special covariance matrix representation allows using corresponding Kalman filter calculation methods. They provide numerical stability of the algorithm. The Kalman approach makes it possible to combine modern techniques used in airborne survey data processing. I give an example of the Kalman filter use in the frequency-domain airborne data processing.

Key words: inversion, Kalman filter, airborne electromagnetics, frequency domain, time domain.

INTRODUCTION

The Kalman filter (KF) is a widely used tool of optimal state estimation theory (Simon, 2006). Though initially it was formulated for linear problems (Kalman, 1960) later the algorithm was extended and applied to nonlinear problems as well. The most KF applications are for the problems related to various dynamic systems, but some quite complicated KF algorithms are applied to ill-posed problems (Keppenne and Rienecker, 2003).

Traditional methods for airborne electromagnetics (AEM) data processing are apparent resistivity calculation and one-dimensional (1D) inversion, since these methods provide a quick representation of physical properties of the environment without any additional information. Apparent resistivity is calculated as the resistivity of equivalent homogeneous half-space for each channel in frequency domain (FD) or in time domain (TD). Generally, the geometry of AEM system, including its altitude above the ground, is used in the calculations. 1D inversion is calculation of parameters for horizontally layered model in a way that the modeled response

corresponds to the measurements of AEM system. The geometry is taken into account also. For both methods it is required to solve problems which are ill-posed according to Hadamard (1932). Thus, a regularization is required in order to provide a unique and stable solution.

According to Chang-Chun et al. (2015) a generally accepted set of methods of 1D inversion exists in the AEM. Guillemoteau et al. (2011) choose among them three basic methods: Vertically Constrained Inversion (VCI), Laterally Constrained Inversion (LCI), Singular Value Decomposition (SVD). All these methods use different techniques to eliminate measurement noise, but the basis of all of them is minimization of the following quadratic function:

$$Q = \sum_{j=1}^{N}[F_j^{obs} - F_j^{cal}(\mathbf{m})]^2, \qquad (1)$$

where N is the number of channels of the AEM system, which are used in the processing; F^{obs} are measurement results and F^{cal} are solutions of the forward problem for the model with parameters given by the vector $\mathbf{m} = (\rho_1, h_1, \rho_2, h_2, ...)$, where ρ_k is the resistivity and h_k is the thickness of the corresponding layer.

According to Legault (2015) the vast majority of modern AEM systems use vertical magnetic dipole to transmit field and vertical component of the response to interpret. Following the trend, for given frequency ω I will calculate function F in formula (1) as the vertical component of the response (Zhdanov, 2009):

$$H_z(\omega) = -\frac{1}{2\pi}\int_0^\infty u(n_0, z, h_T, \omega)J_0(n_0 r)n_0^2 dn_0, \qquad (2)$$

where J_0 is zero order Bessel function of the first kind, r is the horizontal shift of the receiver with respect to the dipole axis, h_T is the altitude of the dipole above ground, u is two-dimensional spectrum of the potential of the secondary field:

$$u(n_0, z, h_T, \omega) = \frac{M \cdot \exp(-n_0(z+h))}{2} \cdot \frac{n_1 - n_0 R^*}{n_1 + n_0 R^*}, \qquad (3)$$

where M is the amplitude of the dipole moment, R^* is the reduced spectral impedance of the medium. For K layers (the thickness of the bottom layer is considered to be infinite) the reduced spectral impedance is given as

$$R^* =$$
$$\text{th}\left\{n_1 h_1 + \text{th}^{-1}\left[\frac{n_1}{n_2}\text{th}\left(n_2 h_2 + \cdots\left(n_{K-1}h_{K-1} + \text{th}^{-1}\frac{n_{K-1}}{n_K}\right)\cdots\right)\right]\right\},$$

$$n_k = \sqrt{n_0^2 - \frac{i\omega\mu_0}{\rho_k}}, \quad \text{Re}\, n_k > 0, \qquad (4)$$

$\mu_0 = 4\pi \cdot 10^{-7}$ is the magnetic constant, i is the imaginary unit, Re n_k is the real part of the complex number n_k.

The equations (2)-(4) are sufficient to solve the forward problem in 1D case and to obtain the function F for AEM FD systems. The real part of (2) gives the in-phase component and the imaginary part gives the quadrature component of secondary field.

To obtain the model of the TD response, it is enough to convolve the frequency response of the layered model $SH_z(\omega)$ with known spectrum of the primary field $ST(\omega)$. The components of the frequency response are calculated setting $M = 1$ in (3). Taking into account that the waveform of all AEM TD systems is periodical and symmetrical with respect to zero and that the receiver frequency response $SR(\omega)$ is bounded at least by the Nyquist frequency, it is sufficient to calculate the response for finite number of odd harmonics of the base frequency (Volkovitsky and Karshakov, 2013):

$$H_z(t) = \frac{1}{2\pi} \mathrm{Re} \sum_{k=0}^{L} SH_z(\omega_k) \cdot ST(\omega_k) \cdot SR(\omega_k) \cdot \exp(-i\omega_k t),$$
$$\omega_k = (2k+1)\omega_0, \tag{5}$$

where ω_0 is the base frequency of the primary field, i. e. the minimal frequency in the waveform spectrum; ω_L is the maximum frequency, that can be registered by the receiver, $SR(\omega_k) = 0$ for $k > L$.

Obviously, setting the number of layers $K = 1$ in (4) leads to the forward problem solution for apparent resistivity. According to Fraser (1987), there are several approaches used to calculate apparent resistivity as a function of two parameters of FD data. 1) In-phase and quadrature part: ineffective for low conductivity – there is no in-phase response, and for high conductivity – there is an ambiguous quadrature response, nonsensitive to resistivity changes. 2) Amplitude and flight altitude: a most appropriate method, but it gives greater noise for low conductivity with respect to method 4, and greater noise for high conductivity with respect to method 3. 3) In-phase part and flight altitude: ineffective for low conductivity – there is no in-phase response. 4) Quadrature part and flight altitude: ineffective for high conductivity – there is an ambiguous quadrature response. 5) Phase and flight altitude: ineffective for low conductivity – phase is nonsensitive to resistivity changes.

Any of these methods provides the use of 2D nomogram for calculating apparent resistivity, which, with the use of modern software packages, allows obtaining the desired values quickly. Each of these approaches has its advantages and disadvantages, but it is worth noting the following. First, all of them give different values of resistivity when the real medium differs from a homogeneous half-space, which makes it difficult to switch from one method to another. And such a transition is necessary, because of the mentioned limitations. Second, the secondary field for a layered medium differs from the response for a half-space. This leads to poor adequacy of real measurements and calculated in-phase and quadrature parts: the measured quadrature part can be reasonably greater than calculated one for any half-space.

In this paper I suggest to use the KF for solution of the following two tasks: first, to combine modern techniques used in 1D inversion, second, to get a general algorithm for apparent resistivity calculation. The rest of the article is organized as follows. First, the inverse problem is formulated as a stochastic estimation problem, it is necessary to apply the KF. For the solution of this problem I give the equations of the iterated extended KF (IEKF) (Havlik and Straka, 2015).

The peculiarity of the proposed approach is the way of calculating the covariance matrix, alternative to the methods described by Havlik and Straka (2015). An emphasis is made on the composition of covariance matrix of estimate error. It is shown, how this algorithm implements known approaches to 1D-inversion in case of VCI, LCI and SVD. As an example, the described algorithm is applied to the data of AEM FD system EM4H (Vovenko et al., 2013) for apparent resistivity calculation. At the last part of the article the conclusions are formulated.

GEOPHYSICAL INVERSION AS A STOCHASTIC ESTIMATION PROBLEM

At each moment j the measurements of a geophysical system can be represented as N-dimensional vector $\mathbf{z}_j$. Assume that the model of medium at that point is described by K-dimensional vector of parameters $\mathbf{x}_j$ and parameters of the field are derived from the solution of the forward problem as N-dimensional vector function $\mathbf{h}_j(\mathbf{x}_j)$:

$$\mathbf{z}_j = \mathbf{h}_j(\mathbf{x}_j) + \mathbf{r}_j, \quad E[\mathbf{r}_j] = 0, \quad E[\mathbf{r}_j \mathbf{r}_k^T] = \mathbf{R}_j \delta_{jk}, \tag{6}$$

where $\mathbf{r}_j$ is a vector of measurement noise, which is modeled as Gaussian white noise with zero mean and covariance matrix $\mathbf{R}_j$, $E[\cdot]$ denotes the mean value, δ_{jk} is the Kronecker delta: $\delta_{jk} = 1$ for $j = k$ and $\delta_{jk} = 0$ otherwise.

Also it is possible to use the model of time dependence of the vector $\mathbf{x}_j$:

$$\mathbf{x}_{j+1} = \mathbf{f}_j(\mathbf{x}_j) + \mathbf{q}_j, \quad E[\mathbf{q}_j] = 0, \quad E[\mathbf{q}_j \mathbf{q}_k^T] = \mathbf{Q}_j \delta_{jk}, \tag{7}$$

where $\mathbf{q}_j$ is a vector of random component that is modeled as Gaussian white noise with zero mean and covariance matrix $\mathbf{Q}_j$. In essence, this matrix represents correlation of medium parameters in neighboring points. Note, the equation (7) can be considered as a spatial dependence of $\mathbf{x}$, i. e. matrix $\mathbf{Q}$ and vector $\mathbf{f}$ can be functions of coordinate along the survey profile (Karshakov and Kharichkin, 2008).

The estimation problem is formulated as follows. Assume that the state vector $\mathbf{x}$ satisfies (7). From the measurements $\mathbf{z}_j$, which satisfy (6), at the moments j it is required to find vectors $\widetilde{\mathbf{x}}_j$ that have minimum deviation from the actual values $\mathbf{x}_j$ at these points.

Initial estimate $\widetilde{\mathbf{x}}_0^-$ of vector of parameters is needed to solve the problem. This estimate does not take any measurements into account and reflects only a priori information about the medium. For this vector covariance matrix of a priori estimate error is used:

$$\widetilde{\mathbf{x}}_0^- = E[\widetilde{\mathbf{x}}_0], \quad \mathbf{P}_0^- = E[(\mathbf{x}_0 - \widetilde{\mathbf{x}}_0^-)(\mathbf{x}_0 - \widetilde{\mathbf{x}}_0^-)^T]. \tag{8}$$

This statement of the estimation problem lets use of the IEKF.

THE ALGORITHM OF ITERATED EXTENDED KALMAN FILTER

The solution of the estimation problem is given by a sequence of cycles. Each of them consists of two steps: the correction step and the prognosis step. The prognosis step is a transition from a posteriori estimates, marked by '+', in point $j - 1$, to a priori estimates, marked by '−', in point j:

$$\widetilde{\mathbf{x}}_j^- = \mathbf{f}_{j-1}(\widetilde{\mathbf{x}}_{j-1}^+),$$

$$\mathbf{P}_j^- = \mathbf{A}_{j-1}\mathbf{P}_{j-1}^+\mathbf{A}_{j-1}^T + \mathbf{Q}_{j-1}, \quad \mathbf{A}_{j-1} = \frac{\partial \mathbf{f}_{j-1}}{\partial \mathbf{x}}, \tag{9}$$

There are several effective methods for severely nonlinear models (7). They take into account nonlinearity of **f** by different ways: sigma-point algorithm, unscented filter etc. (Simon, 2006). However, the deterministic part **f** in (7) is usually unknown, because our knowledge about the medium is minimal. All the uncertainty is put into the random part **q**. So, in most cases the following model can be used: $\mathbf{f}(\mathbf{x}) = \mathbf{x}$, $\mathbf{A} = \mathbf{I}$.

The correction step is a transition from a priori estimate to a posteriori one at the moment *j*:

$$\tilde{\mathbf{x}}_j^{k+} = \tilde{\mathbf{x}}_j^{k-} + \mathbf{K}_j^k\big(\mathbf{z}_j - \mathbf{h}_j(\tilde{\mathbf{x}}_j^{k-})\big),$$

$$\mathbf{P}_j^{k+} = (\mathbf{I} - \mathbf{K}_j^k\mathbf{H}_j^k)\mathbf{P}_j^{k-},$$

$$\mathbf{K}_j^k = \mathbf{P}_j^{k-}\mathbf{H}_j^{kT}[\mathbf{H}_j^k\mathbf{P}_j^{k-}\mathbf{H}_j^{kT} + \mathbf{R}_j]^{-1}, \quad \mathbf{H}_j^k = \frac{\partial \mathbf{h}_j(\tilde{\mathbf{x}}_j^{k-})}{\partial \mathbf{x}}. \quad (10)$$

The function **h(x)** in (10) is severely nonlinear – see (2) or (5), for example. That is why several iterations are needed during the correction step. The upper index *k* in equations (10) designates the iteration number. During each step the Jacobian matrix **H** and filter coefficient **K** should be recalculated. The breaking condition for the iteration procedure is met when the following value achieves its minimum:

$$\big\|\mathbf{z}_j - \mathbf{h}_j(\tilde{\mathbf{x}}_j^{k+})\big\| = \sqrt{\big(\mathbf{z}_j - \mathbf{h}_j(\tilde{\mathbf{x}}_j^{k+})\big)^T \mathbf{R}^{-1}\big(\mathbf{z}_j - \mathbf{h}_j(\tilde{\mathbf{x}}_j^{k+})\big)}. \quad (11)$$

If the measurement noise is Gaussian, (11) corresponds to the maximum a posteriori estimate – maximum of the corresponding distribution function (Simon, 2006).

During each iteration step the following operations should be made:

$$\tilde{\mathbf{x}}_j^{k-} = \tilde{\mathbf{x}}_j^{k-1+}, \quad \mathbf{P}_j^{k-} = \frac{\big\|\mathbf{z}_j - \mathbf{h}_j(\tilde{\mathbf{x}}_j^{k-1+})\big\|^2}{\big\|\mathbf{z}_j - \mathbf{h}_j(\tilde{\mathbf{x}}_j^{k-1-})\big\|^2}\mathbf{P}_j^{k-1-}. \quad (12)$$

This empirical formula for the recalculation of the covariance matrix **P**, not common for the IEKF, avoids too fast reduction of its norm to zero, which is the result of the fact that nonlinearity is not taken into account in equations (10) for **P**.

ONE DIMENSIONAL INVERSION

All the formalism presented above can be applied to various inversion problems that are not necessarily related to AEM. Further I will consider the details of 1D inversion in case of AEM survey.

First of all, let's remember that rock's resistivity follows a log normal distribution (Palacky, 1987), so I will consider **x** as logarithm of **m** from equation (1).

In case of AEM survey matrix **R** can be obtained from calibration flight data. Such flights are performed at high altitudes, as a rule, at 500-700 m above the ground, so the response is negligible (Vovenko et al., 2013). The statistic characteristics of the signals completely describe the matrix **R**. The simplest way is to set it as a diagonal matrix, the diagonal elements are the signal dispersions for corresponding channel.

As a matter of fact, the matrix **Q** describes possible limits of model parameters variation from one measurement point to another: its diagonal elements are the dispersions of increment/decrement of corresponding parameters. It is clear that for greater distance between measurement points the greater variability of the model is required. Thus, it is reasonable to set the variability per meter by the matrix $\mathbf{Q}^1$ and calculate matrix $\mathbf{Q}_j$ as $\mathbf{Q}_j = v_j^2\mathbf{Q}^1$, where v_j is the magnitude of the current velocity of the carrier of the AEM system. And the model itself can be trivial $\mathbf{f}_j(\mathbf{x}_j) = \mathbf{x}_j$.

Finally, the matrix $\mathbf{P}_0^-$, which characterizes the adequacy of a priori information about the model, should contain the dispersions of the errors for corresponding parameters in the main diagonal and their correlation coefficients outside it.

Discussing the methods of 1D inversion it should be noted that the form (10) in general is equivalent to the traditional forms for SVD, VCI, LCI (Jupp and Vozoff, 1975; Guillemoteau et al., 2011). It is important that in case of VCI and LCI the thickness of each layer is fixed.

According to Guillemoteau et al. (2011) VCI method suggests using a matrix with non-diagonal elements as a stabilizer. In terms of KF it is equivalent to setting non-diagonal elements in matrix $\mathbf{P}^-$, which are correlations of resistivity of layers. LCI method suggests using a diagonal matrix as a stabilizer. In terms of KF it is equivalent to use of prognosis step, where the matrix $\mathbf{Q}^1$ represents the variability of resistivity per meter, as it was mentioned above. In case of LCI it is reasonable to apply Kalman smoothing algorithm (Simon, 2006). The main idea is to apply one KF from the beginning to the end of the profile under processing, another KF from the end to beginning, and then to combine forward and backward estimates according to their covariance.

For KF numerical stability matrix **P** is usually represented by its Cholesky or LDL decomposition: $\mathbf{P} = \mathbf{S}\mathbf{S}^T$ or $\mathbf{P} = \mathbf{L}\mathbf{D}\mathbf{L}^T$, where **S** is lower triangular matrix, or square root of **P**, **L** is lower unit triangular matrix, **D** is diagonal matrix. Simon (2006) describes the KF algorithm for these forms of **P**. Also he describes the methods of reducing the order of filter, which is equivalent to SVD method.

APPARENT RESISTIVITY CALCULATION

As an example I consider calculation of the apparent resistivity or resistivity of equivalent homogeneous half-space for FD AEM data of the system EM4H (Vovenko at al., 2103). In this system the vertical magnetic dipole is fixed on a fuselage of an aircraft, three-component receiver is towed by a flexible 70 meters long cable. The system transmits an alternating magnetic field in four frequencies (130, 520, 2080, 8330 Hz) and measures the in-phase and quadrature components of the response. Altitude of an aircraft is controlled by a radar altimeter. In addition, the system measures the position of the receiver with respect to the transmitter. As a result, the sum of receiver and transmitter altitudes $(z + h)$ and horizontal shift of the receiver are obtained, which are required in formula (3) to calculate the secondary field (2).

It is required to calculate the resistivity of equivalent homogeneous half-space for each frequency in each point where in-phase (real) and quadrature (imaginary) components of the response are measured. I suggest to apply IEKF having in-phase and quadrature components in each frequency as measurements, sum of receiver and transmitter altitudes $(z + h)$ and horizontal shift of the receiver *r* as parameters in forward problem solution. In this case the dimension of vector of parameters **x** is 1 and it contains the logarithm of

apparent resistivity only. The vector of measurements **z** has dimension 2 and contains the in-phase and quadrature components of the response.

Figure 1 shows the result of IEKF in comparison with the resistivity calculated with the nomogram method on the base of quadrature response and flight altitude. It is clear, that the inversion with IEKF provides the solution uniqueness and the efficiency for different resistivity values. And for the nomogram method there are serious difficulties in 2 and 8 kHz frequencies. The quadrature component of the response is close to the maximum value and has an ambiguity and a weak dependence on resistivity.
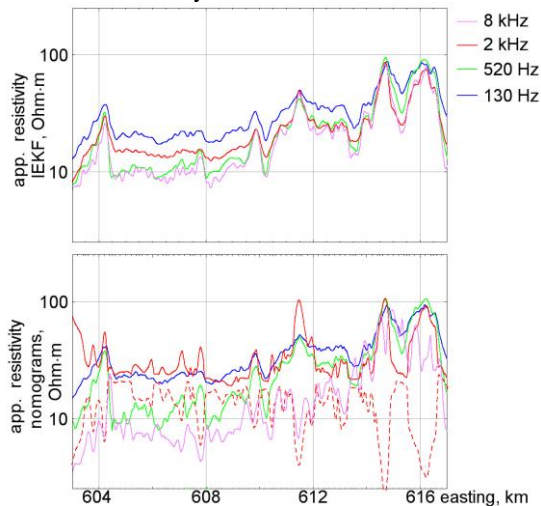

Figure 1. EM4H data inversion, from up to down: IEKF apparent resistivity, Ohm·m; apparent resistivity for quadrature response and altitude nomograms, Ohm·m. Dashed red line corresponds to conductive nomogram for 2 kHz and solid red line – to resistive nomogram for 2 kHz.

CONCLUSIONS

The inversion problem for AEM is an essential part of data processing. Variety of the methods even for 1D case is available. Sometimes the choice of the most appropriate method may be quite sophisticated. The method described in this abstract considers AEM data inversion as a stochastic estimation problem and uses well-developed tools of estimation theory based on KF. As a result, it allows combining the modern approaches to inversion and provide efficient solution for various types of medium. The results of the algorithm application are shown by the example of one of the simplest problem – the apparent resistivity calculation for FD AEM system EM4H. The algorithm automatically estimates the influence of each component of the measurement vector. It provides smooth and unique solution for conditions, where traditional methods have difficulties. Besides, the method takes into account the geometry of the system. It allows avoiding flight restrictions: substantial variations of flight speed or even variations of flight altitude in a certain diapason do not lead to substantial distortions of obtained apparent resistivity.

REFERENCES

Chang-Chun, Y., Xiu-Yan, R., Yun-He, L., Yan-Fu, Q., Chang-Kai, Q. and Jing, C., 2015, Review on airborne electromagnetic inverse theory and applications: Geophysics, 80(4), W17-W31.

Fraser, D.C., 1987, Layered-earth resistivity mapping: Fitterman, D.V. (Ed.), Developments and Applications of Modern Airborne Electromagnetic Surveys, US Geological Survey Bulletin, 33-41.

Guillemoteau, J., Sailhac, P. and Béhaegel, M., 2011, Regularization strategy for the layered inversion of airborne transient electromagnetic data: application to in-loop data acquired over the basin of Franceville (Gabon): Geophysical Prospecting, 59, 1132-1143.

Hadamard, J., 1932, Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques: Hermann.

Havlik, J. and Straka, O., 2015, Performance evaluation of iterated extended Kalman filter with variable step-length: Journal of Physics: Conference Series, 659, 012-022.

Jupp, D.L.B. and Vozoff, K., 1975, Stable iterative methods for the inversion of geophysical data: Geophysical Journal of the Royal Astronomical Society, 42, 957-976.

Kalman, R., 1960, A new approach to linear filtering and prediction problems: ASME Journal of Basic Engineering, 82, 35-45.

Karshakov, E.V. and Kharichkin, M.V., 2008, A stochastic estimation problem at aeromagnetometer deviation compensation: Automation and Remote Control, 69(7), 1162-1170.

Keppenne, C.L. and Rienecker, M., 2003, Assimilation of temperature into an isopycnal ocean general circulation model using a parallel Ensemble Kalman filter: Journal of Marine Systems, 40-41, 363-380.

Legault, J.M., 2015, Airborne electromagnetic systems – state of the art and future directions: CSEG Recorder, 40(6), 38-49.

Palacky, G.J., 1987, Resistivity characteristics of geologic targets: Nabighian, N.M., (Ed.) Electromagnetic Methods, SEG, 53-129.

Simon, D., 2006, Optimal State Estimation. Kalman, H∞ and Nonlinear Approaches: John Wiley & Sons, Inc., Hoboken, New Jersey.

Volkovitsky, A. and Karshakov, E., 2013, Airborne EM systems variety: what is the difference?: Papers of the 13th SAGA Biennial @ 6th International AEM Conference AEM-2013. Mpumalanga, South Africa, 1-4.

Vovenko, T., Moilanen, E., Volkovitsky, A. and Karshakov, E., 2013, New abilities of quadrature EM systems: Papers of the 13th SAGA Biennial @ 6th International AEM Conference AEM-2013. Mpumalanga, South Africa, 1-4.

Zhdanov, M.S., 2009, Geophysical Electromagnetic Theory and Methods: Elsevier.